Topological Entropy in Condensed Matter Physics

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Disambiguation

Topological Entropy in CMT $\neq$ Topological entropy in mathematics

In mathematics:

the measure of complexity of the topological dynamic system (Adler, Konheim, McAndrew 1965)

TE is the average amount of information required to describe a map on Hausdorff space.

Historical reasons of introducing the topological order

Experimental results

High-T_c superconductors (Müller, Bednorz 1986, Nobel Prize 1987), $T_c = 110K$.

Theoretical approach

Chiral spin state. Attempts to describe by:

1. Landau symmetry-breaking theory - chiral spin state breaking time reversal and parity, but not spin rotation symmetry
2. topological order (from TQFT) - need for new quantum numbers:
   - ground state degeneracy
   - geometric phase (non-Abelian)

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   - **topological entropy**

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Theories that can be described by TO:

- Quantum Hall effect (TO predicts fractional quantum Hall states - discovery in 1982)
- topological insulator boundary states
- $\mathbb{Z}_2$ superconductors

Schrödinger's cat

Description

- cat $\mapsto |\text{alive}\rangle$ or $|\text{dead}\rangle$,
- trap $\mapsto |\text{trap} - \text{off}\rangle$ or $|\text{trap} - \text{on}\rangle$.

State after "interaction"

$$|\text{cat} - \text{trap}\rangle = \frac{1}{\sqrt{2}} \left(|\text{alive}\rangle \, |\text{trap} - \text{off}\rangle + |\text{dead}\rangle \, |\text{trap} - \text{on}\rangle \right) \quad (1)$$

Quantum Entanglement

Features

- State of the system is better described than states of its components.
- Non-intuitive - does not translate to the classical world.

Pure and Mixed States

Product states

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be states on different Hilbert spaces H_1 and H_2. A product state:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \,, \tag{2}$$

belongs to:

$$H = H_1 \otimes H_2. \tag{3}$$

Pure and mixed states

If $|\mathbf{e}_1\rangle$ and $|\mathbf{e}_2\rangle$ are basis in H_1, H_2 respectively, states in a form of:

$$|\psi\rangle = \sum_{a \in \mathbf{e}_1, b \in \mathbf{e}_2} c_{a,b} |\psi_{a,b}\rangle \quad (4)$$

(where $|\psi_{a,b}\rangle = |a\rangle \otimes |b\rangle$), are called pure states if

$$c_{a,b} = c_a c_b. \quad (5)$$

Otherwise they are called mixed (or entangled) states.

Density matrix

Introduction

For an operator $\mathbb{A}$ given, its average can be written as:

$$\langle \mathbb{A} \rangle = \sum_{i_1, i_2} c_{i_1} c_{i_2}^* \langle \psi_{i_2} | \mathbb{A} | \psi_{i_1} \rangle \tag{6}$$

Assuming form of c_i coefficient in as:

$$c_{i_1} c_{i_2}^* = \langle \psi_{i_1} | \rho | \psi_{i_2} \rangle , \tag{7}$$

is defining a matrix ρ, called *density matrix*:

$$\rho = \sum_i w_i | \psi_i \rangle \langle \psi_i | , \tag{8}$$

Thus, the averaging reads as:

$$\langle \mathbb{A} \rangle = \mathrm{Tr}\left(\rho \mathbb{A}\right). \tag{9}$$

Reduced density matrix

Going back to discussed product space $H_1 \otimes H_2$ we can project density matrix on one of component spaces:

$$\rho_1 = \sum_{b \in \mathbf{e}_2} \langle b | \, \psi \rangle \langle \psi | \, b \rangle \,. \tag{10}$$

Outcome

- the density matrix ρ describes the system
- a new mathematical framework with explicit method of calculating averages

Reduced density matrices for Schrödinger's cat w. fun.

Trap trace out

$$\rho_1 = \frac{1}{2} \langle t - off |\, t - off \rangle \, |alive\rangle \langle alive| \, \langle t - off |\, t - off \rangle + \frac{1}{2} \langle t - on |\, t - on \rangle \, |dead\rangle \langle dead| \, \langle t - on |\, t - on \rangle \quad (11)$$

Cat trace out

$$\rho_2 = \frac{1}{2} \langle alive |\, alive \rangle \, |t - off\rangle \langle t - off| \, \langle alive |\, alive \rangle + \frac{1}{2} \langle dead |\, dead \rangle \, |t - on\rangle \langle t - on| \, \langle dead |\, dead \rangle \quad (12)$$

Reduced density matrices for Schrödinger's cat w. fun.

$$\rho_1 = \rho_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

von Neumann Entropy

Definition

$$S(\rho) = -\mathrm{Tr}(\rho \ln \rho) \tag{14}$$

Properties

- $S(\rho) = 0$ for pure state
- $S(\rho)_{max} = \log(D)$ for so called "maximally mixed state"
- $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$

Schrödinger's cat

$$S(\rho_{cat-trap}) = S(\rho_1) + (\rho_2) = \log 2 + \log 2 = 2 \log 2 \tag{15}$$

Concept

An universal characterisation of the many-particle quantum entanglement in the ground state of topologically ordered 2-dim medium.

A general description of topological order for 2-dim planes.

Levin-Wen Definition

$$-S_{topo} = (S_1 - S_2) - (S_3 - S_4), \quad (16)$$

where S_1, S_2, S_3, S_4 are von Neumann entropies for A_1, A_2, A_3, A_4 respectively.

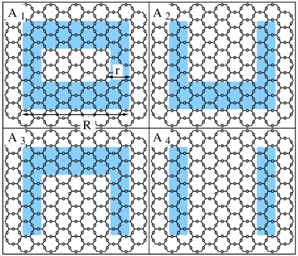

Figure: Note that A_1 differs from A_2 in the same way as A_3 from A_4. *Figure from [3]*

Kitaev-Preskill Definition

Problem description

For a ball, with boundary length L, on a 2-plane given, the expresion for von Neumann entropy is:

$$S = \alpha L - \gamma + \ldots, \tag{17}$$

where terms vanishing for $L \to \infty$ are ellipsed, α is a coefficient depending on wave function behaviour around the boundary, and γ is a nonnegative, universal, additive constant.

Kitaev-Preskill Definition

$$S_{topo} = S_A + S_B + S_C + S_{ABC} - S_{AB} - S_{BC} - S_{CA} = \gamma \qquad (18)$$

where S_{AB} is von Neumann entropy for a area as on figure below.

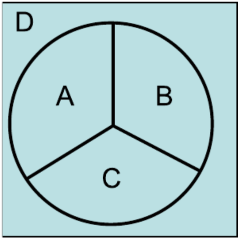

Figure: $ABC = A \cup B \cup C$ is much larger than corelation length. *Figure from [2]*

Topological invariance

Let's consider a small deformation of discussed ball on a boundary between regions C and D. Since it does not change A nor B:

$$\Delta S_{topo} = (\Delta S_{ABC} - \Delta S_{BC}) - (\Delta S_{AC} - \Delta S_{C}). \tag{19}$$

If concidered regions are much bigger than corelation length including A in concidered regions will give a neglectable change.

Topological invariance

The three-point deformation (as seen on a figure):

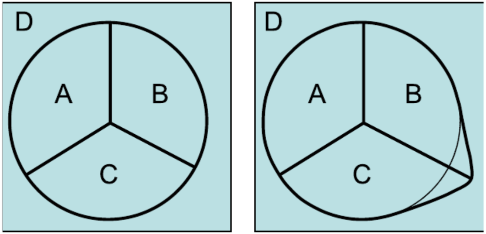

Figure: *Figure from [2]*

Topological invariance

Can be described as:

$$\Delta S_{topo} = (\Delta S_B - \Delta S_{AB}) + (\Delta S_C - \Delta S_{AC}) - (\Delta S_{ABC} - \Delta S_{BC}). \tag{20}$$

Which analogicaly disappears.

Universality

Invariance under deformation of Hamiltonian

Assuming the Hamiltonian is a sum of local terms:

1. if deformation is far from the boundary, it does not change the entropy
2. if deformation is close to the boundary, the topological invariance can be used to "move" the deformation far from the boundary

Remarks

- Both definitions are equivalent.
- The topological entropy is:
  - invariant under unitary transformations.
  - universal.
  - a topological invariant.
- Description of topological order on the level of wave functions.

Applications

1. quantum computers
   - choosing materials
   - reducing decoherence (information lasts longer)
   - toric code - quantum error handling
2. classification of trial wave functions
3. topological insulators

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Thank you very much for your attention!

Are there any questions?