Jagiellonian University
Department of Physics, Astronomy and Applied Informatics

Evolution of a massless test scalar field on Boson Stars space-times

Author:
Andrzej P. Kądzielawa

Under supervision of:
dr Andrzej Rostworowski

Kraków 2011

Contents

Chapter 1

Introduction

From all hypothetical astronomical objects the most famous and well-studied are Black Holes. The simplest mathematical model of the Black Hole is described by Schwarzschield metric. The solution of Einstein equations:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + d\Omega^2, \qquad (1.0.1)$$

is given in terms of mass M.

Less recognisable is a specific class of objects called the Black Hole Mimickers - the alternative solutions in general relativity that can exist as observed Black Hole Candidates (for example *Sagittarius A** in the center of the Milky Way [EGA$^+$05]). Among them are Gravastars, Wormholes, Brane World solutions and Boson Stars.

Lora-Clavijo, Cruz-Osorio and Guzmán in [LCCOG10a] examined the model of Boson Stars towards finding the characteristics distinguishing Boson Stars from Schwarzschield Black Holes. They obtained Boson Star space-time, compactified it and evolve massless scalar fields on obtained background. They claimed that a distinguishing characteristics might be a tail of massless scalar field decay - t^{-p} for Black Holes (p is constant parameter), but not for Boson Stars.

The massless scalar field decay in Boson Star space-time should decay in the same way as Schwarzschield space-time.

Our goal is to redo the calculations, find Boson Star metric, compactify it and find the evolution of massless scalar field on obtained space-time to compare with results of [LCCOG10a].

Chapter 2

Boson Stars

In Chapter 2 the mathematical model of Boson Star will be explained. The Lagrangian density with stress-energy tensor it implies will be given. The metric Ansatz will be proposed. System of equations resulting from Einstein equations and Einstein-Klein-Gordon system of equations will be shown.

Boson Star is a configuration of self-gravitating complex scalar field in equilibrium. It was first introduced by Kaup [Kau68] and Ruffini and Bonazzola [RB69], as a spherically symmetric solution to Einstein-Klein-Gordon equations. Friedberg, Lee and Pang [FLP87] found in 1987 that, unlike in Ruffini's and Bonazzola's work, there is more than one solution to a given system of equations, and a countable series of the solutions were introduced.

Unlike other Black Hole Mimickers the stability of Boson Stars was well examined [Gle88], making them potentially valuable objects to examine.

2.1 Mathematical assumptions

Let us take a complex scalar field Φ (with its complex conjugate Φ^*) and the Lagrangian density as follows:

$$\mathcal{L} = -\frac{R}{\kappa_0} + g^{\mu\nu}\partial_{mu}\Phi^*\partial_\nu\Phi + V\left(|\Phi|^2\right), \tag{2.1.1}$$

where $\kappa_0 = 16\pi G$ (assuming $c = 1$), and V is the potential of self-interaction of the field. The stress-energy tensor reads:

$$T_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\Phi^*\partial_\nu\Phi + \partial_\mu\Phi\partial_\nu\Phi^*\right) - \frac{1}{2}g_{\mu\nu}\left(\partial^\alpha\Phi^*\partial_\alpha\Phi + V\left(|\Phi|^2\right)\right). \tag{2.1.2}$$

For Boson Stars we will use the potential V in a form:

$$V = m^2|\Phi|^2 + \frac{\lambda}{2}|\Phi|^4, \tag{2.1.3}$$

where m is to be considered as mass of a single boson, and λ as a coefficient of a two-body self-interaction mean field approximation.
Additionally the function $\Phi(t,r)$ can be separated into:

$$\Phi(t,r) = e^{i\omega t}\phi(r). \tag{2.1.4}$$

The sketch of proof of such a separation can be found in Appendix B.

From (2.1.2) and (2.1.3) we can easily show that stress-energy tensor (for Φ as in (2.1.4)) does not depend on time. The Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa_0 T_{\mu\nu} \tag{2.1.5}$$

brings us to the conclusion that the geometry of the space-time is time-independent.

2.1.1 Einstein-Klein-Gordon equation

The Bianchi identity can be reduced to the Klein-Gordon equation:

$$\left(\Box - \frac{dV}{d\left|\Phi\right|^2} \right) \Phi = 0, \tag{2.1.6}$$

where $\Box . = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu . \right)$, and g is determinant of the metric. Solutions of (2.1.6) with conditions set above are called Boson Stars.

2.2 Metric ansatz and equations

To obtain the metric we must introduce a metric Ansatz. Let it be:

$$ds^2 = -\alpha\left(r\right)^2 dt^2 + a\left(r\right)^2 dr^2 + r^2 d\Omega^2. \tag{2.2.1}$$

We will derive the equations for $\partial_r \alpha(r)$ and $\partial_r a(r)$ from (2.1.5):

$$\frac{\partial_r a(r)}{a(r)} = \frac{1 - a(r)^2}{2r} + \frac{r}{2} \frac{a(r)^2}{\alpha(r)^2} T_{tt}, \tag{2.2.2a}$$

$$\frac{\partial_r \alpha(r)}{\alpha(r)} = \frac{a(r)^2 - 1}{2r} + \frac{r}{2} T_{rr}. \tag{2.2.2b}$$

Using (2.2.2) and (2.1.2), we can obtain explicit form of the equations, but with the third function $\Phi(t, r)$. To complete the system of equations we will need one more equation - (2.1.6). $\Phi(t, r)$ can be separated as in (2.1.4). After the substitution the system of equations reads:

$$\frac{\partial_r a}{a} = \frac{1-a^2}{2r} + \frac{\kappa_0 r}{4}\left(\omega^2\phi^2\frac{a^2}{\alpha^2} + (\partial_r\phi)^2 + a^2\phi^2\left(m^2 + \frac{\lambda}{2}\phi^2\right)\right), \quad (2.2.3a)$$

$$\frac{\partial_r \alpha}{\alpha} = \frac{a^2-1}{r} + \frac{\partial_r a}{a} - \frac{\kappa_0 r}{2}a^2\phi^2\left(m^2 + \frac{\lambda}{2}\phi^2\right), \quad (2.2.3b)$$

$$\partial_{rr}\phi + \partial_r\phi\left(\frac{r}{2} + \frac{\partial_r\alpha}{\alpha} - \frac{\partial_r a}{a}\right) + \omega^2\phi\frac{a^2}{\alpha^2} - a^2\phi\left(m^2 + \lambda\phi^2\right) = 0. \quad (2.2.3c)$$

System (2.2.3) contains three parameters and one constant, but this number can be reduced to just one by introducing new variables. We rescale the old ones to obtain the minimal number of parameters in equations:

$$\hat{\phi} = \sqrt{\frac{\kappa_0}{2}}\phi, \quad (2.2.4a)$$

$$\hat{r} = mr, \quad (2.2.4b)$$

$$\hat{\alpha} = \frac{m}{\omega}\alpha, \quad (2.2.4c)$$

$$\hat{t} = \omega t, \quad (2.2.4d)$$

$$\Lambda = \frac{2\lambda}{\kappa_0 m^2}. \quad (2.2.4e)$$

Where $\hat{t}$ and $\hat{r}$ are dimensionless, constant κ_0 disappears from equations, and only one parameter - Λ - is left. After removing the hats:

$$\frac{\partial_r a}{a} = \frac{1-a^2}{r} + \frac{r}{2}\left[\phi^2\frac{a^2}{\alpha^2} + (\partial_r\phi)^2 + a^2\left(\phi^2 + \frac{\Lambda}{2}\phi^4\right)\right], \quad (2.2.5a)$$

$$\frac{\partial_r \alpha}{\alpha} = \frac{a^2-1}{r} + \frac{\partial_r a}{a} - ra^2\phi^2\left(1 + \frac{\Lambda}{2}\phi^2\right), \quad (2.2.5b)$$

$$\partial_{rr}\phi + \partial_r\phi\left(\frac{2}{r} + \frac{\partial_r\alpha}{\alpha} - \frac{\partial_r a}{a}\right) + \phi\frac{a^2}{\alpha^2} - a^2\left(1 + \Lambda\phi^2\right)\phi = 0. \quad (2.2.5c)$$

2.2.1 Metric function behaviour estimation for small radius

To integrate system of equations (2.2.5) numerically, we need values of the functions in point of origin. Additionally we require regularity of system of equations at r (particularly at $r = 0$), and asymptotic flatness of metric for $r \to \infty$.

It is crucial to examine the behaviour of the functions $a(r)$, $\alpha(r)$ and $\phi(r)$ around zero, because, as we can see in (2.2.5), there are terms $\propto r^{-1}$, that can not be solved numerically naïve way. The results are:

- $a(r = 0) = 1$,
- $\phi(r = 0)$ is finite,
- $\alpha(r = 0)$ is finite,
- $\partial_r \phi(r = 0) = 0$.

Asymptotic flatness gives $\phi(r \to \infty) = 0$.

To obtain those results we expand those functions around 0 in terms of Taylor series, then puts expansions into the equations and disentangles the expressions for each coefficient. This leads to the form:

$$a(r) = 1 + a_2 r^2 + a_4 r^4 + a_6 r^6 + o(r^8), \tag{2.2.6a}$$

$$\alpha(r) = \alpha_0 + \alpha_2 r^2 + \alpha_4 r^4 + \alpha_6 r^6 + o(r^8), \tag{2.2.6b}$$

$$\phi(r) = \varphi_0 + \varphi_2 r^2 + \varphi_4 r^4 + \varphi_6 r^6 + o(r^8). \tag{2.2.6c}$$

All coefficients are uniquely determined by three parameters $\alpha(0)$, $\phi(0)$ and Λ. Except for the important values of derivatives of given functions at $r = 0$ (obviously 0) and the second derivative of $\phi(r)$ (the r^2 term times 2), we obtain important information about metric functions α and a and field amplitude ϕ - they are even functions around $r = 0$ with accuracy $o(r^8)$. Explicit forms of coefficients can be found in Appendix A.1.

2.2.2 Sketch of solution

Friedberg, Lee and Pang in [FLP87] showed that there are more than one solution to Boson Star metric equations. We can identify those solutions, creating from them the countable set, numbered by the number of zeros. ϕ_0 will be the ground state (with no zeros) and ϕ_n the excited states (n - number of zeros).

We have initial conditions for $a(r=0)=1$ and asymptotic value $\phi_0(r \to \infty) \to 0$. For numerical solution we need value of either $\alpha(0)$ or $\phi_0(0)$ (the other must be left free to the shooting method to satisfy the value of ϕ_0 in infinity).

We solve system of equations (2.2.5) with Runge-Kutta fourth-order method with $\alpha(0)$ as a shooting parameter with the respect to two parameters - $\phi_0(0)$ and Λ. Because of our interest in the ground state, the shooting method must be slightly modified: we add the condition of solution having no zeros.

The figure 2.1 represents first step of bisection (described in Section 6.1)- searching the interval of initial value $[\alpha(0)_L, \alpha(0)_R]$, such that for $\alpha(0)_L = 0.88194$ $\phi(r)$ has one zero and for $\alpha(0)_R = 0.88472$ $\phi(r)$ has no zeros. On the figure 2.2 are plots of the solutions for the initial value of $\alpha(r)$ $\alpha(0)_L, \alpha(0)_R$ and the solution $\phi_0(r)$ from the second step of bisection.

2.2.3 Mass function $M(r)$

For Schwarzschield space-time $a(r) = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$. Analogically we can introduce mass function $M(r)$. From (2.2.5a):

$$M(r) = \frac{r}{2}\left(1 - \frac{1}{a(r)^2}\right). \tag{2.2.7}$$

It is obvious that to ensure asymptotic flatness at infinity we must have finite limit of $M(r \to \infty) = M_\infty < +\infty$. In other words for big r, then function $\phi(r) \xrightarrow{r\to\infty} 0$, and $a(r)$ is equal to $\left(1 - \frac{2M_\infty}{r}\right)^{-\frac{1}{2}}$. It is consistent with metric asymptotics obtained in [BW00]. Equation (2.2.5b) for changes into:

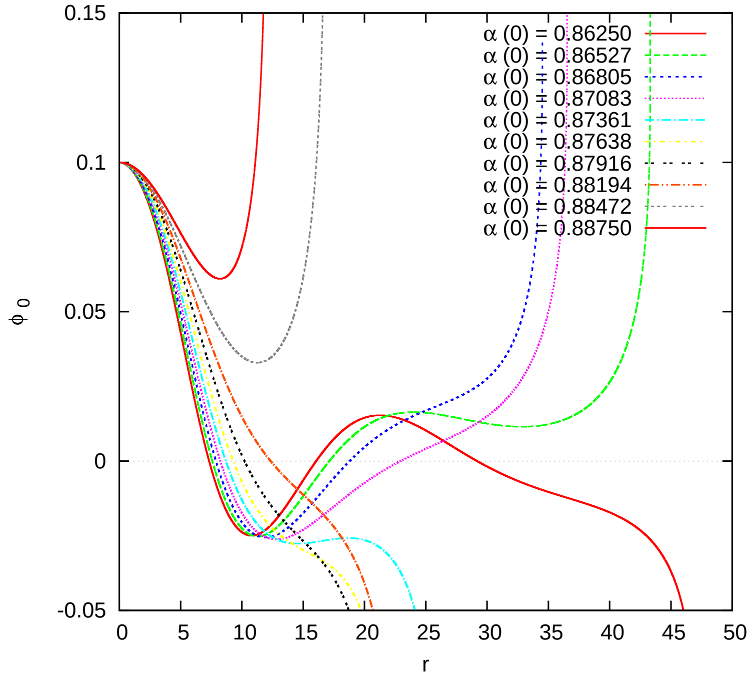

Figure 2.1: Plots of solutions for $\phi_0(r)$ for different values of $\alpha(0)$. It can be easily spotted that solution must be between $\alpha(0) = 0.88194$ and $\alpha(0) = 0.88472$.

$$\frac{\partial_r \alpha}{\alpha} = \frac{M_\infty}{r^2} \frac{1}{1 - \frac{2M_\infty}{r}} + o\left(\phi^2\right). \tag{2.2.8}$$

That can be integrated and simplify to:

$$\alpha = \sqrt{1 - \frac{2M_\infty}{r}}. \tag{2.2.9}$$

We can observe on the figure 2.3 how M_∞ depends on parameters - Λ and $\phi(0)$.

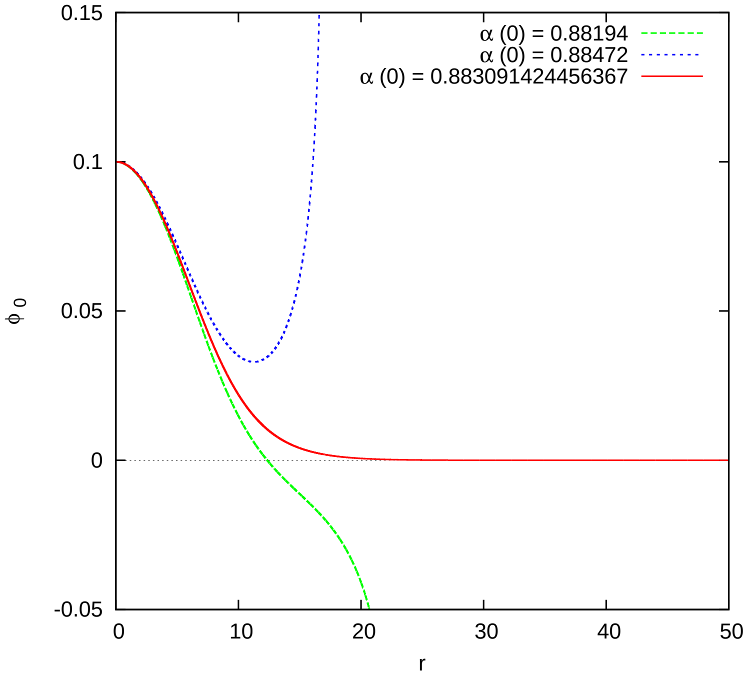

Figure 2.2: Plot of solution $\phi_0(r)$ (red), and solutions for initial borders of bisection interval.

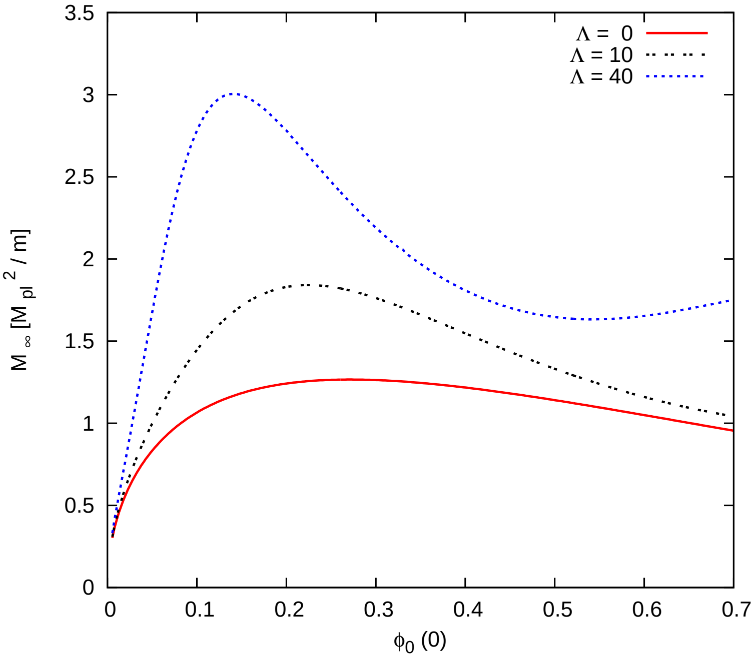

Figure 2.3: Mass function $M(r)$ value in infinity (given in units from (2.2.5)) for different values of Λ as a function of central value $\phi(0)$.

Chapter 3

Hyperboloidal foliations and scri-fixing

In Chapter 3 3+1 formalism [Gou07] will be briefly discussed, the hyperboloidal foliations will be introduced, and it's numerical form will be shown. Method of compactification will be introduced. The behaviour of height function's derivative around $\hat{r} = 0$ will be shown.

In numerical methods of solving partial differential equations (like wave equation in 1+1 dimensions) we may encounter an obstacle: For infinite domain we are forced to introduce artificial border in one of the coordinates. After finite interval in time, the signal from the border will influence solution in whole domain, thus the border must be simulate in the proper way.

It is not always clear how neglect influence of the border. We may try to avoid this by compactifying naïvely the spatial coordinate:

$$\hat{r} \in [0, +\infty] \to r \in [0, 1]. \tag{3.0.1}$$

Let us consider (following [Zen10]) the oscillating function $\psi(t, \hat{r})$:

$$\psi(t, \hat{r}) = e^{2\pi i (k\hat{r} - \omega t)}, \tag{3.0.2}$$

where $\hat{r}$ is spatial coordinate $\hat{r} \in [0, +\infty], k \in \mathbb{R}, \omega \in \mathbb{R}$. Since ψ is oscillating infinite number of times in the direction of $\hat{r}$, after compactification to r we will loose information about oscillations.

We can omit this difficulty by first changing time coordinate such that in the direction of the new one ψ is oscillating finite number of times:

$$\tau(t, \hat{r}) = t - \frac{k}{\omega}\left(\hat{r} + \frac{C}{1+\hat{r}}\right). \tag{3.0.3}$$

Then (3.0.2) changes into:

$$\psi(\tau, \hat{r}) = e^{2\pi i \left(\frac{kC}{1+\hat{r}} - \omega\tau\right)}, \tag{3.0.4}$$

that has maximum kC oscillations (for $\hat{r} \in \{kC-1, \frac{kC}{2}-1, \dots, \frac{kC}{[kC]}-1\}$).

The transformation for space-time is much more complicated, but idea is the same.

3.1 Hyperboloidal foliations

Let the metric be in a form as given in (2.2.1):

$$ds^2 = -\alpha\left(\hat{r}\right)^2 dt^2 + a\left(\hat{r}\right)^2 d\hat{r}^2 + \hat{r}^2 d\Omega^2. \tag{3.1.1}$$

The time coordinate t slices space-time into Cauchy hypersurfaces, approaching spatial infinity in the asymptotic region. We are interested in the null infinity, therefore we need the new coordinate $\tau = \tau(t, \hat{r})$, slicing space-time into hyperboloidal hypersurfaces.

The important feature of stationary metric is natural representation of *Killing observers*. A set of points (observers) travelling in the direction of Killing vector field will not distort distances from each other. Then, the Killing observers would be a good detector of gravitational waves (hence the relative movement between each of them can not be originated from space-time curvature), and leaving it in an particularly easy form (given by ∂_t) is considered important.

$$\partial_\tau = \frac{\partial t}{\partial \tau}\partial_t + \frac{\partial r}{\partial \tau}\partial \hat{r},$$

$$\text{and from above: } \partial_\tau = \partial_t,$$

$$\text{which leads to: } \frac{\partial t}{\partial \tau} = 1 \text{ and } \frac{\partial \hat{r}}{\partial \tau} = 0.$$

The restriction above brings the expression on τ to:

$$\tau = t - h(\hat{r}), \tag{3.1.2}$$

where sign is a convention, and $h(\hat{r})$ will be referred to as *height function.* In terms of new coordinates the metric will carry a new form:

$$ds^2 = -\alpha\left(\hat{r}\right)^2 d\tau^2 - 2\alpha\left(\hat{r}\right)^2 h'\left(\hat{r}\right) d\tau d\hat{r} + \left(-\alpha\left(\hat{r}\right)^2 h'\left(\hat{r}\right)^2 + a\left(\hat{r}\right)^2\right) d\hat{r}^2 + \hat{r}^2 d\Omega^2, \tag{3.1.3}$$

where $h'\left(\hat{r}\right) = \frac{dh}{d\hat{r}}$. The choice of height function is still not clear. We require more information about its behaviour. In the next sections one of the methods will be shown.

3.2 Metric compactification

For computational purposes, we require metric to be compactified (as shown at the beginning of this chapter). By introducing:

$$r = r(\hat{r}), \tag{3.2.1}$$

where r is restricted to finite interval. At the boundary of the domain the metric (3.1.3) becomes singular. The singularity is removed by multiplying the metric by conformal factor Ω^2 (not to be misunderstood as infinitesimal part of $\mathcal{S}^2$ metric will be written as $d\hat{\Omega}$), that vanishes at the boundary. Furthermore we require that the new coordinate r is an area radius on the new metric which implies:

$$\hat{r}(r) = \frac{r}{\Omega}. \tag{3.2.2}$$

We obtain:

$$\begin{aligned} ds^2 = & -\Omega^2 \alpha\left(r\right)^2 d\tau^2 - 2\alpha\left(r\right)^2 h'\left(r\right)\left(\Omega - r\Omega'\right) d\tau dr + \\ & + \frac{\left(-\alpha\left(r\right)^2 h'\left(r\right)^2 + a\left(r\right)^2\right)}{\Omega^2}\left(\Omega - r\Omega'\right)^2 dr^2 + r^2 d\hat{\Omega}^2, \end{aligned} \tag{3.2.3}$$

where $\Omega' = \frac{d\Omega}{dr}$. The choice of Ω is free, although there are some forms that proved to be useful for different metrics. For example for Schwarzschield space-time we usually choose $\Omega = 1 - r$. For Boson Stars the convenient conformal factor is:

$$\Omega = \frac{1 - r^2}{2}, \tag{3.2.4}$$

compactifying metric on $[0, 1]$ interval.

3.3 Heights function for Boson Stars space-time in 3+1 formalism

Obtaining the direct expression for $h(t)$, even though on the first moment desired, is not a necessity - in the compactified metric there is only the derivative of height function. Therefore we should look for expression for $h'(t) = \frac{dh}{dr}$. For this purpose we will use the properties of $3+1$ formalism and the (3.2.3) metric written in such a form. Before calculation the brief explanation of functions used in model will be provided.

3.3.1 3+1 formalism

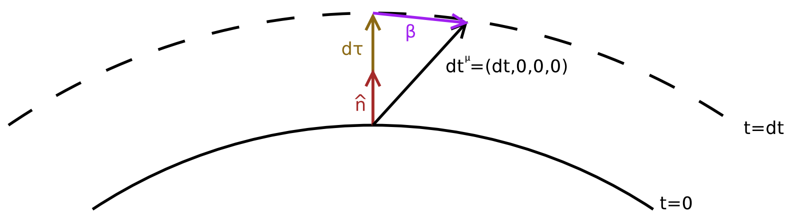

Figure 3.1: The diagram of elapse of proper time τ and time t

Let's consider the case, where time-space is described by metric $g^{\mu\nu}$ and time-constant, space-like hypersurface Σ_t by metric (3 dimensional) γ_{ij}. Let $\hat{n}$ be a time-like vector, normal to hypersurface, and $V^\mu = (0, V^1, V^2, V^3)$ be a vector tangent to hypersurface. We can show that:

$$\hat{n} \cdot V = 0 \Longrightarrow n_\mu = (-\alpha, 0, 0, 0) \longrightarrow n^\mu = -\alpha(g^{tt}, g^{ti}) \tag{3.3.1}$$

where $\alpha > 0$ functions is called laps. We demand $\hat{n}$ to be normal, thus:

$$-1 = n_\mu n^\mu = \alpha^2 g^{tt}. \tag{3.3.2}$$

We can introduce the projection of dt on $\hat{n}$, as on figure 3.1:

$$d\tau = \frac{(\hat{n}, (dt^\mu))}{(\hat{n}, \hat{n})}\hat{n} = \alpha dt \hat{n}, \tag{3.3.3}$$

where $(\cdot,\cdot)$ is scalar product. Furthermore:

$$d\tau = \alpha dt\hat{n} \longrightarrow d\tau^{\mu} = -\alpha^2 dt(g^{tt}, g^{ti}). \tag{3.3.4}$$

Shift β:

$$d\tau + \beta = dt \rightarrow \beta = dt - d\tau \longrightarrow \beta^{\mu} = dt\left(1 + \alpha^2 g^{tt}, \alpha^2 g^{ti}\right). \tag{3.3.5}$$

$\beta^t = 0$, because we demanded $\hat{n}$ normal.

From (3.3.1), (3.3.4) and (3.3.5) we can read off the form of metric $g^{\mu\nu}$ (remembering that on Σ_t the metric is γ_{ij}:

$$g_{\mu\nu} = \left[\begin{array}{c|c} -\alpha^2 + \beta^k\beta_k & \beta_i \\ \hline \beta_i & \gamma_{ij} \end{array}\right] \tag{3.3.6}$$

and

$$g^{\mu\nu} = \left[\begin{array}{c|c} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \hline \frac{\beta^i}{\alpha^2} & \gamma_{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{array}\right] \tag{3.3.7}$$

Expression (3.3.6) can be rewritten in a form:

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right). \tag{3.3.8}$$

Since we are interested in spherically symmetric space-time above equation can be rewritten as:

$$ds^2 = \left(-\alpha^2 + \gamma^2\beta^2\right)dt^2 + 2\beta\gamma^2 dr dt + \gamma^2 dr^2 + r^2 d\Omega^2, \tag{3.3.9}$$

were Ω is standard metric on $\mathcal{S}^2$, β and γ are scalar function in new coordinates.

In Riemann Geometry the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ carries the information of curvature of a manifold. Distinguishing a hypersurface in manifold makes a similar construction for hypersurface desirable. We introduce (for time-constant hypersurface in space-time) a new object carrying information about curvature of Σ_t. Form:

$$\mathbf{k} : \Sigma_t \times \Sigma_t \to \mathbb{R}, \tag{3.3.10a}$$

$$\mathbf{k}\,(u, v) = -u \cdot \nabla_v n \tag{3.3.10b}$$

will be called *extrinsic curvature*. It is evaluating the change of direction of normal vector 'moving' on hypersurface. In analogy to Ricci scalar we can ask about scalar value with information about curvature. Consider:

$$k = Tr\mathbf{k} = \gamma^{ij} k_{ij} \tag{3.3.11}$$

- *the mean extrinsic curvature.* By (3.3.10b), the expression for mean extrinsic curvature will be given by:

$$k = \nabla n = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} n^\mu \right), \tag{3.3.12}$$

where g is determinant of the metric.

3.3.2 Derivation of $h'(t)$ equation for Boson Stars

We can observe the similarity of its form and a form of 3+1 metric (as shown in (3.3.9) with bars added to avoid misapprehensions):

$$d\bar{s}^2 = \left(-\bar{\alpha}^2 + \bar{\gamma}^2 \bar{\beta}^2 \right) dt^2 + 2\bar{\beta}\bar{\gamma}^2 d\hat{r} dt + \bar{\gamma}^2 d\hat{r}^2 + \hat{r}^2 d\Omega^2 \tag{3.3.13}$$

and a non-compactified space-time, with the transformation of time variable. After the transformation constant time hypersurfaces are hyperboloidal slices reaching $\mathcal{J}^+$:

$$ds^2 = -\alpha\,(\hat{r})^2\, d\tau^2 - 2\alpha\,(\hat{r})^2\, h'\,(\hat{r})\, d\tau d\hat{r} + \left(-\alpha\,(\hat{r})^2\, h'\,(\hat{r})^2 + a\,(\hat{r})^2 \right) d\hat{r}^2 + \hat{r}^2 d\Omega^2. \tag{3.3.14}$$

If we desire to have metric written in such a form, we should disentangle the $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ functions:

$$\bar{\alpha}(\hat{r}) = \frac{\alpha(\hat{r})a(\hat{r})}{\bar{\gamma}(\hat{r})}, \tag{3.3.15a}$$

$$\bar{\beta}(\hat{r}) = -\frac{h'(\hat{r}\alpha(\hat{r})^2)}{\bar{\gamma}(\hat{r})^2}, \tag{3.3.15b}$$

$$\bar{\gamma}(\hat{r})^2 = a(\hat{r})^2 - \alpha(\hat{r})^2 h'(\hat{r})^2. \tag{3.3.15c}$$

Having the explicit form of those functions we can use the properties of $3+1$ formalism to obtain the expression for the mean extrinsic curvature k. To proceed we need the form of the unit normal vector to the spatial hypersurfaces pointing to the future (by equation (3.3.1)):

$$[n^\mu] = \left[\frac{\bar{\gamma}}{\alpha a}, \frac{h'\alpha}{\bar{\gamma} a}, 0, 0\right]. \tag{3.3.16}$$

Now, using (3.3.12):

$$k = \frac{1}{\hat{r}^2 \alpha(\hat{r}) a(\hat{r})} \partial_{\hat{r}} \left(\frac{\hat{r}^2 h'(\hat{r})\alpha(\hat{r})}{\bar{\gamma}(\hat{r})}\right). \tag{3.3.17}$$

For constant k we can integrate equation above with integration constant $-\frac{C}{k}$:

$$k \int \hat{r}^2 \alpha(\hat{r}) a(\hat{r}) d\hat{r} - C = \frac{\hat{r}^2 h'(\hat{r})\alpha(\hat{r})}{\sqrt{a(\hat{r})^2 - \alpha(\hat{r})^2 h'(\hat{r})^2}}. \tag{3.3.18}$$

From (3.3.18) we can disentangle expression for $h'(r)$:

$$h'(\hat{r}) = \frac{\left(kI(\hat{r}) - C\right) a(\hat{r})}{\alpha(\hat{r})\sqrt{\left(kI(\hat{r}) - C\right)^2 + \alpha(\hat{r})^2 \hat{r}^4}} \tag{3.3.19}$$

where

$$I(\hat{r}) = \int \hat{r}^2 a(\hat{r})\alpha(\hat{r}) d\hat{r}. \tag{3.3.20}$$

Equation for $h(r)$ can be obtained by integrating equation (3.3.19). Expression in such a form can not be solved easily (especially that we do not know the analytical forms of metric functions for Boson Stars) using methods others than numerical. Nevertheless we do not need to solve such an equation, because the compactified metric functions does depend only on $h'(t)$. The equation (3.3.19) depends on two parameters, which are not given - k and C (except for course of Λ and $\phi_0(0)$ that determine solution of $\alpha(\hat{r})$ and

$a(\hat{r})$). Choice of values of those parameters seems to be free, but there are some restrictions.
$k < 0$ implies that normal vector to hypersurface Σ_t is pointing into past, which would bring to non-physical results. k then must be positive. Later we will obtain other restrictions for C. k we will choose (following [LCCOG10a]) equal to 3.

3.3.3 h'(r) behaviour around r=0

We expand (3.3.19) in power series, and compare coefficients we can obtain behaviour of $h'(r)$ around $r = 0$:

1. $C \neq 0$
$$h'(r) = -\frac{1}{\alpha_0(0)} + h_2 r^2 + h_4 r^4 + O(r)^6, \tag{3.3.21}$$
2. $C = 0$
$$h'(r) = \frac{k}{12\alpha(0)} r + \bar{h}_3 r^3 + o(r^5), \tag{3.3.22}$$

where h_2 and h_4 are rather complicated in its form rational functions of k, C, $\phi_0(0)$, $\alpha(0)$ and Λ and can be found in Appendix A.2. This formula is useful in further studies of behaviour of background functions in section 4.1.

3.4 Metric compactification

Degree of complication of equations obtained leads to necessity of numerical approach. Infinite domain causes trouble on choice of the cut-off radius, therefore the compactified domain with finite size is most desirable for computational methods. Following (3.2.1) compactification of coordinate $\hat{r}$ and rescaling the metric by a conformal factor Ω (not to be misunderstood as metric on $\mathcal{S}^2$), leads to metric form (3.2.3):

$$\begin{aligned} ds^2 = & -\Omega^2 \alpha(r)^2 d\tau^2 - 2\alpha(r)^2 h'(r)(\Omega - r\Omega') d\tau dr + \\ & + \frac{\left(-\alpha(r)^2 h'(r)^2 + a(r)^2\right)}{\Omega^2} (\Omega - r\Omega')^2 dr^2 + r^2 d\hat{\Omega}^2. \end{aligned} \tag{3.4.1}$$

Expression above is general, the choice of conformal factor Ω is restricted by regularity of Ricci scalar. Following [BZ09], in case of Boson stars, we choose:

$$\begin{aligned} \Omega &= \frac{1-r^2}{2}, \\ \hat{r}(r) = \frac{r}{\Omega} &= \frac{2r}{1-r^2}, \end{aligned} \tag{3.4.2}$$

that was proved to have regular Ricci scalar in whole domain.

To conclude we can list the steps needed to obtain compactified metric for Boson Stars:

1. Solve the set of equations (2.2.5), using shooting method to ensure $\phi_0(\hat{r} \to +\infty) = 0$ for big-but-finite $\hat{r}$.

2. Fit 'mass' function (2.2.7) to last points calculated for $a(\hat{r})$.

3. Match Schwarzschield solution for higher $\hat{r}$.

4. Find values of $h'(\hat{r})$ from (3.3.19).

5. Compactify metric by (3.4.1) for conformal factor $\Omega = \frac{1-r^2}{2}$.

The figure 3.2 shows solutions of $a(\hat{r})$, $\alpha(\hat{r})$, $h'(\hat{r})$ and $g^{tt}(r)$ (compactified space-time), calculated for $\Lambda = 20$, $\phi_0(0) = 0.1$, $k = 3$, $C = 0$.

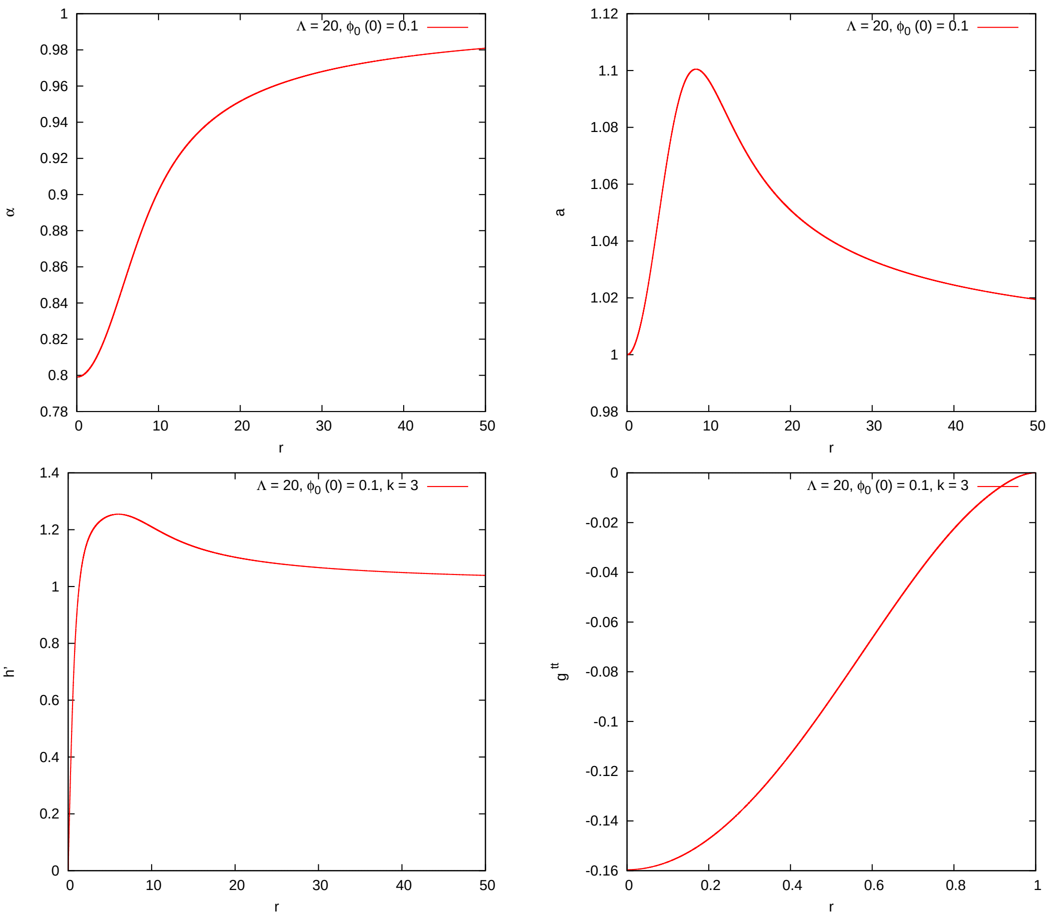

Figure 3.2: **A** - metric function for time for Boson Star space-time; **B** - metric function for radius for Boson Star space-time; **C** - height function's derivative; **D** - g^{tt} element for metric of compactified Boson Star space-time; All calculation are made for parameter values: $\Lambda = 20$, $\phi_0(0) = 0.1$. The height function's derivative and metric compactification are computed with $k = 3$ and $C = 0$. Obtained parameter $\alpha(0) = 0.7990477629$

Chapter 4

Wave equation

In Chapter 4 the system of equations equivalent to wave equation will be proposed. Regularity of background function in the system of equations will be examined. Candidates for initial values of function will be proposed. The results of numerical calculation will be discussed.

The wave equation in any non-compactified space-time is given by:

$$\hat{\Box}_g \hat{\phi} = 0, \tag{4.0.1}$$

where $\tilde{\Box}_g \cdot = \nabla_\mu \nabla^\mu \cdot = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \cdot)$.

After compactification we obtained metric useful for numerical computation, but not physical - what we are interested in is evolution of scalar field ϕ_T on non-compactified space-time (3.3.14). Such goal can be achieved by finding the form of equation (4.0.1) that its solution will be scaled function $\phi_T(t,r) = \Omega \hat{\phi}_T(t,\hat{r})$. Following [Zen10] we can write:

$$\left(\Box_g - \frac{R}{6}\right) \phi_T(t,r,\theta,\varphi) = \Omega^{-3} \left(\hat{\Box}_g - \frac{\hat{R}}{6}\right) \hat{\phi}_T(t,r,\theta,\varphi) = 0, \tag{4.0.2}$$

where R and $\hat{R}$ are Ricci scalars for compactified and non-compactified metric respectively. As we can see (4.0.2) in non-compactified space-time has a term proportional to Ricci scalar (which do not take place in (4.0.1)).

For Schwarzschield space-time it has no difference (since $\hat{R}_{\text{Schwarzschield}} = 0$) for solutions. For Boson Stars it will not influence on behaviour of $\hat{\phi}_T$ for $t \to \infty$, which is all we are interested in.

To simplify, using spherical symmetry we can substitute: $\phi_T(t,r,\theta,\varphi) = \phi(t,r) Y_{l,m}(\theta,\varphi)$, where $Y_{l,m}(\theta,\varphi)$ are spherical harmonics.

4.1 Splitting into set of equations

First we will one more time use the similarity of obtained metric (3.2.3) and the one describing the general case of 3+1 model. This time we will find the formulas of laps function $\breve{\alpha}$, shift function $\breve{\beta}$ and function $\breve{\gamma}$ (that is what was left form induced metric γ_{ij} after applying spherical symmetry condition) in a form analogical to (3.3.13):

$$\breve{\gamma}(r)^2 = \frac{\left(a(r)^2 - \alpha(r)^2 h'(r)^2\right)\left(\Omega - r\partial_r \Omega\right)^2}{\Omega^2}, \tag{4.1.1a}$$

$$\breve{\beta}(r) = -\frac{\alpha(r)^2 h'(r)\left(\Omega - r\partial_r\Omega\right)}{\breve{\gamma}(r)^2}, \tag{4.1.1b}$$

$$\breve{\alpha}(r)^2 = \alpha(r)^2\Omega^2 + \breve{\beta}(r)^2\breve{\gamma}(r)^2. \tag{4.1.1c}$$

We will refer to the functions above as *background functions*. It is not obvious that those functions are regular (in particularly, since $h'(0) = \frac{1}{\alpha(0)}$, the function $\breve{\gamma}(r) \xrightarrow{r\to 0} 0$ and both $\breve{\alpha}(r)^2$ and $\breve{\beta}(r)$ are proportional to $\frac{1}{\breve{\gamma}(r)^2}$. We will study the potential points where poles may appear in subsection 4.1.1.

System of equations (4.0.2) contain terms $\propto \partial_\mu\partial_\nu\cdot$ therefore we need new variables (functions of time and radius) for numerical computation. We introduce:

$$\pi(t,r) = \frac{\breve{\gamma}}{\breve{\alpha}}\partial_t\phi - \frac{\breve{\gamma}}{\breve{\alpha}}\breve{\beta}\partial_r\phi, \tag{4.1.2a}$$

$$\psi(t,r) = \partial_r\phi. \tag{4.1.2b}$$

Expressions with background functions $\breve{\alpha}$, $\breve{\beta}$, $\breve{\gamma}$ might be in general singular - there is no rule forbidding them to be infinite or 0 in some points. We could compensate such poles with proper behaviour of dynamical fields, but it would be easier to find such a set of parameters, that those expressions are regular everywhere.

4.1.1 Study of regularity of background functions

We can find two of those points. $r = 0$ where $\breve{\gamma} = 0$ (for $C \neq 0$), and $r = 1$. In the second part we have $\lim_{r\to 1}\breve{\gamma} = \frac{0}{0}$, because: $\Omega = \frac{1-1^2}{2} = 0$ and $a(r)^2 - h'(r)^2\alpha(r)^2 = 1^2 - 1^2 1^2 = 0$.

$r = 0$

We have already found the Power series of $\alpha(\hat{r})$ and $a(\hat{r})$ (Appendix A.1) and $h'(\hat{r})$ (Appendix A.2). Knowing that:

$$\Omega = \frac{1-r^2}{2},$$

$$\Omega - r\partial_r\Omega = \frac{1+r^2}{2},$$

we can find expansion of background functions:

$$\begin{aligned}\breve{\alpha}(r) = &\frac{C^2}{64r^4} - \frac{3C^2}{32r^2} - \frac{Ck\alpha(0)}{12r} + \left(\frac{15C^2}{64} + \frac{\alpha(0)^2}{4}\right) + \\ &+ \frac{Ck\left(5\alpha(0)^2 - 2\phi_0(0)^2\right)r}{20\alpha(0)} + o(r^2),\end{aligned} \tag{4.1.4}$$

$$\begin{aligned}\breve{\beta}(r) = &\frac{C^2}{32\alpha(0)r^4} - \frac{C^2(5\alpha(0)^2 + 2\phi_0(0)^2)}{32\alpha(0)^3r^2} - \frac{Ck}{6r} + \\ &+ \frac{1}{144\alpha(0)^5}(36\alpha(0)^6 - C^2(\text{Polynomial}(\phi_0(0), \alpha(0), \Lambda))+ \\ &+ \frac{1}{15}Ck(5 + \frac{2\phi_0(0)^2}{\alpha(0)^2})r + o(r^2),\end{aligned} \tag{4.1.5}$$

$$\breve{\gamma}(r) = o(r^4). \tag{4.1.6}$$

Equations (4.1.4) and (4.1.5) are singular in $r = 0$ for every constant $C \neq 0$. For this model only $C = 0$ integration constant can give regular background.

For $C = 0$:

$$\breve{\alpha}(r) = \frac{\alpha(0)^2}{4} + o(r^2), \tag{4.1.7}$$

$$\breve{\beta}(r) = -\frac{1}{3}(k\alpha(0))r + o(r^3), \tag{4.1.8}$$

$$\breve{\gamma}(r) = 1 + \left(-\frac{4k^2}{9} + \frac{2\phi_0(0)^2\left(2 + \alpha(0)^2\left(2 + \Lambda\phi_0(0)^2\right)\right)}{3\alpha(0)^2}\right)r^2 + o(r^4). \tag{4.1.9}$$

all of the functions are regular.

In (4.1.19) there is one more complex function $\frac{\breve{\alpha}}{\breve{\gamma}}$ that might have pole for $C \neq 0$ in $r = 0$ (since $\breve{\gamma}(0) = 0$). Since for $C = 0$, $\gamma(0) = 1$ we assured regularity of such function in r=0. Nevertheless let us name it $\Xi(r)$:

$$\Xi(r) = \frac{\alpha(0)^2}{4} + \frac{1}{18}\left(6\phi_0(0)^2 + \alpha(0)^2\left(4k^2 - 3\left(3 + 4\phi_0(0)^2 + 2\Lambda\phi_0(0)^4\right)\right)\right) r^2 + o(r^4). \tag{4.1.10}$$

$\Xi(r)$ is regular in $r = 0$ and it's first derivative disappears in that point. Similarly we can define function $\Upsilon = \breve{\alpha}\breve{\gamma}$:

$$\frac{\alpha(0)^2}{4} + \left(-\frac{\alpha(0)^2}{2} + \phi_0(0)^2\right) r^2 + o(r^4), \tag{4.1.11}$$

regular in $r = 0$ as well.

To examine one more important feature of functions $\breve{\beta}$, Ξ and Υ we calculated them with bigger accuracy (metric function with $o(r^14)$), which gave us an important conclusion:

Around $r = 0$ function $\breve{\beta}$ is odd and functions Ξ and Υ are even.

which will allow us to estimate its numerical derivative around $r = 0$.

$r = 1$

It is much easier to look for poles in this case, because we can calculate the explicit analytical form of those functions using properties of Boson Star space-times (for $r \to +\infty$ space-time goes to Schwarzschield space-time) and explicit expression for function $h'(\hat{r})$ (from (3.3.19)), where $\alpha(\hat{r}) = \sqrt{1 - \frac{2M}{\hat{r}}}$, $a(\hat{r}) = \frac{1}{\alpha(\hat{r})}$ and $I(\hat{r}) = \frac{\hat{r}^3}{3}$:

$$\begin{aligned} h'(\hat{r}) &= \frac{(kI(\hat{r}) - C)\, a(\hat{r})}{\alpha(\hat{r})\sqrt{(kI(\hat{r}) - C)^2 + \alpha(\hat{r})^2\hat{r}^4}} \\ &= \frac{2\left(-C - \frac{8k\hat{r}^3}{3(-1+\hat{r}^2)^3}\right)}{\left(2 + 2M\left(-\frac{1}{\hat{r}} + \hat{r}\right)\right)\sqrt{\left(C + \frac{8k\hat{r}^3}{3(-1+\hat{r}^2)^3}\right)^2 + \frac{8\hat{r}^3(2\hat{r}+2M(-1+\hat{r}^2))}{(-1+\hat{r}^2)^4}}}. \end{aligned} \tag{4.1.12}$$

Similarly we can find explicit forms of all of background functions showing $\breve{\gamma}(r) \xrightarrow{r\to 1} constant$:

Expression for functions are given by:

$$\breve{\alpha}(r) = \sqrt{\frac{9(2M)\left(-1+r^2\right)^3 + 2r\left(9+2\left(-9+2k^2\right)r^2+9r^4\right)}{72r}}, \tag{4.1.13a}$$

$$\breve{\beta}(r) = \frac{-k\sqrt{r^3\left(9(2M)\left(-1+r^2\right)^3 + 2r\left(9+2\left(-9+2k^2\right)r^2+9r^4\right)\right)}}{9\sqrt{2}\left(r+r^3\right)}, \tag{4.1.13b}$$

$$\breve{\gamma}(r) = \sqrt{\frac{18r\left(1+r^2\right)^2}{9(2M)\left(-1+r^2\right)^3 + 2r\left(9+2\left(-9+2k^2\right)r^2+9r^4\right)}}. \tag{4.1.13c}$$

With values in $r = 1$:

$$\breve{\alpha}(1) = \frac{k}{3},$$

$$\breve{\beta}(1) = -\frac{k^2}{9},$$

$$\breve{\gamma}(1) = \frac{3}{k}.$$

4.1.2 Ricci scalar

Since system of equations (4.1.19) contains part proportional to Ricci Scalar, we should find it's form. Out of definition Ricci scalar is:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\rho}_{\mu\rho\nu}, \tag{4.1.15}$$

where $R_{\mu\nu}$ is Ricci tensor and $R^{\rho}_{\mu\lambda\nu}$ Riemann curvature tensor. Expression for Ricci scalar can be found using metric Ansatz (here (3.4.1)) by directly deriving Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. Calculations were done in *Mathematica 8.0*, using *diffgeo.m* library:

$$
\begin{aligned}
R = & \frac{2}{r^2 a(r)^3 \alpha(r)(-\Omega(r) + r\Omega'(r))^3}(-a(r)^3\alpha(r)(\Omega(r) - r\Omega'(r))^3 + \\
& + r\Omega(r)a'(r)(-\Omega(r) + r\Omega'(r))(r\Omega(r)\alpha'(r) + \alpha(r)(2\Omega(r) + r\Omega'(r))) + \\
& + a(r)(\alpha(r)(\Omega(r)^3 - 3r^2\Omega(r)\Omega'(r)^2 - r^3\Omega'(r)^3 + \\
& + 3r\Omega(r)^2(\Omega'(r) + r\Omega''(r))) + r\Omega(r)(-3r^2\alpha'(r)\Omega'(r)^2 + \\
& + \Omega(r)^2(2\alpha'(r) + r\alpha''(r)) + r\Omega(r)(-r\Omega'(r)\alpha''(r) + \\
& + \alpha'(r)(\Omega'(r) + r\Omega''(r)))))),
\end{aligned}
\tag{4.1.16}
$$

where $F'(r) = \frac{dF}{dr}$ and $F''(r) = \frac{d^2F}{dr^2}$ for any function $F \in \{a, \alpha, \Omega\}$. For $r \xrightarrow{\hat{r}\to\infty} 1$ when space-time goes to Schwarzschield-like we can simplify expression (4.1.16):

$$
R = -\frac{12(-1 + r^2)(r(3 + r^2) + 2M(-1 + 2r^2 + r^4))}{r(1 + r^2)^3}, \tag{4.1.17}
$$

and

$$
R(r = 1) = 0. \tag{4.1.18}
$$

Ricci scalar will be regular in whole domain, because of our choice of conformal factor Ω [LCCOG10a].

4.1.3 System of equations

Equation (4.0.2) with angular term separation and new variables as above will change into system of equations:

$$
\partial_t \pi = \frac{1}{r^2}\partial_r\left(r^2\left(\breve{\beta}\pi + \frac{\breve{\alpha}}{\breve{\gamma}}\psi\right)\right) - \breve{\alpha}\breve{\gamma}\left(\frac{1}{6}R\phi + \frac{l(l+1)}{r^2}\phi\right), \tag{4.1.19a}
$$

$$
\partial_t \phi = \frac{\breve{\alpha}}{\breve{\gamma}}\pi - \breve{\beta}\phi, \tag{4.1.19b}
$$

where (4.1.19b) is just rewritten (4.1.2a). We solve system above numerically using method of lines with fourth-order accurate stencils and fourth-order Runge-Kutta method along time coordinate.

4.2 Solution

Figure 4.1 shows background function as discussed in previous sections.

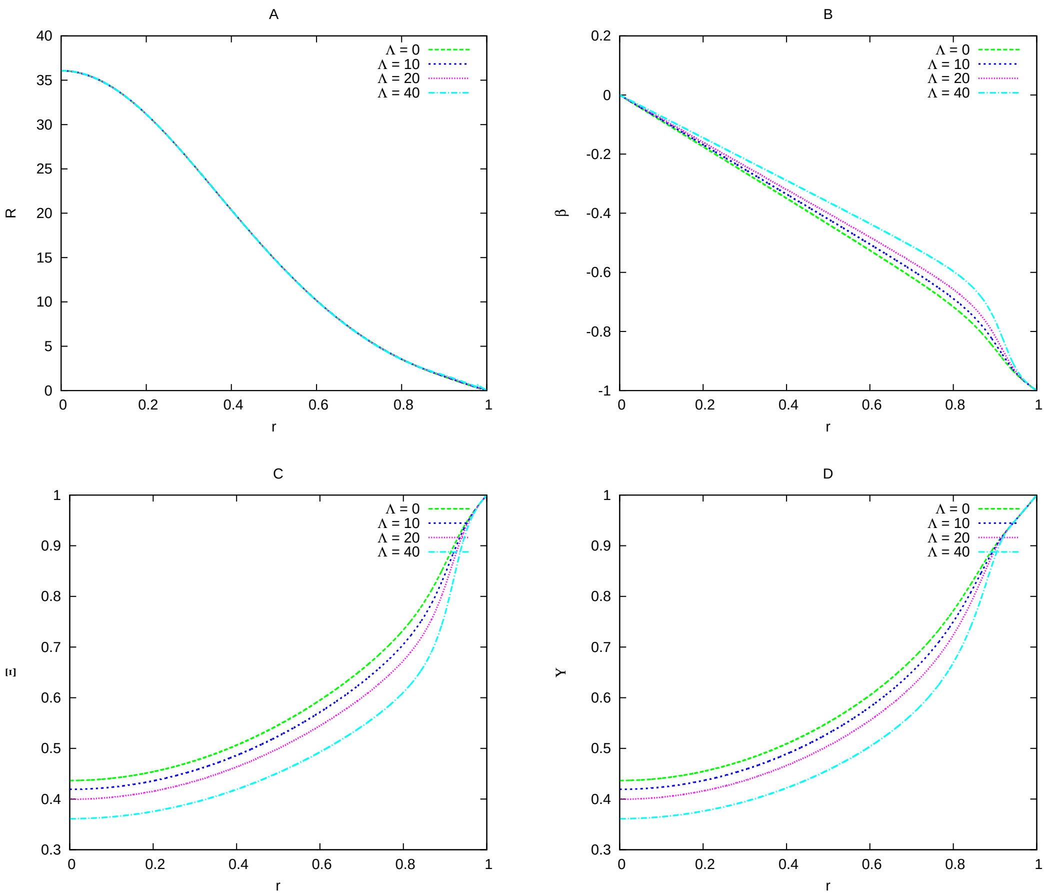

Figure 4.1: **A** - Ricci scalar; **B** - background function $\breve{\beta}$; **C** - background function Ξ; **D** - background function Υ; Calculated for parameters: $\Lambda \in \{0, 10, 20, 40\}$, $\phi_0(0) = 0.1$, $k = 3$, $C = 0$.

We attempt to solve system of equations (4.1.19) with different initial conditions:

$$\phi(r, 0) = 0, \qquad (4.2.1a)$$

$$\psi(r, 0) = 0, \qquad (4.2.1b)$$

$$\pi(r,0) = Ae^{-\frac{\tan{(r \arctan 1)}^2}{\sigma^2}}. \tag{4.2.1c}$$

It differs from approach in [LCCOG10a].

Firstly the system of equations (4.1.19) contains evolution of fields π and ϕ (field ψ is derived from (4.1.2b)). Lora-Clavijo, Cruz-Osorio and Guzmán evolve fields ψ and pi and field ϕ derived afterwards.

Secondly the initial data we used is different from one in [LCCOG10a].

Due to lack of time we did not finished calculations. We obtained first results, that suggest that method we used returns stable outcome. In close future we hope to finish calculations and obtain the answers to the tail decay problem.

Chapter 5

Conclusions

We have obtained the metric functions $a(r)$ and $\alpha(r)$ of Boson Star for given parameters. They are shown on figure 5.1

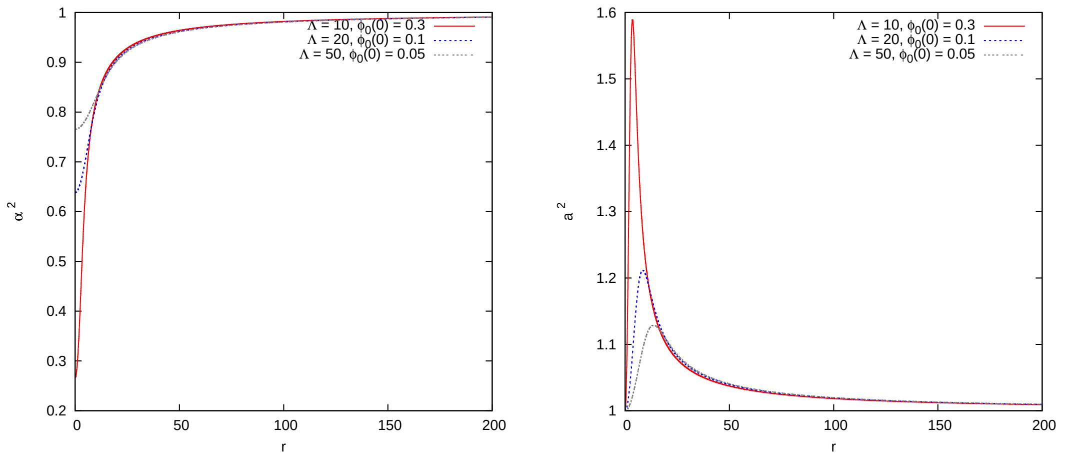

Figure 5.1: Metric functions $\alpha(\hat{r})^2$ (left plot) and $a(\hat{r})^2$ (right plot) for parameters as written on the plots.

They fit to expected Schwarzschield metric for large $\hat{r}$. The height functions derivative was found (on figure 5.2), matching it's equivalent in Schwarzschield metric, as well as background functions for wave equation (4.1.19).

All those functions match with their Schwarzschield equivalents for large $\hat{r}$.

Initial conditions (4.2.1) differs from ones used in [LCCOG10b], [LC-

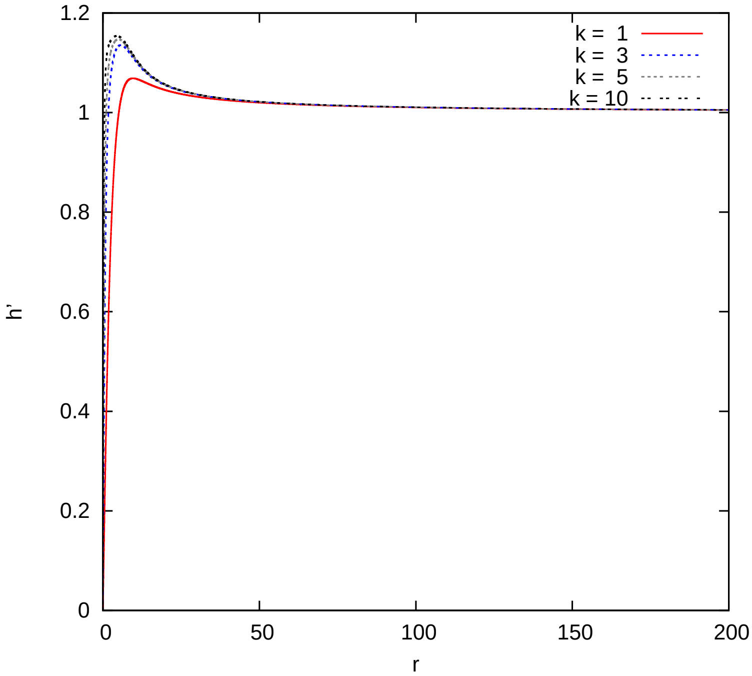

Figure 5.2: First derivative of height function of different k curvature, for $\Lambda = 0$, $\phi_0(0) = 0.1$ and $C = 0$.

COG10a]:

$$\phi(r, 0) = Ae^{-\frac{r^2}{\sigma^2}}, \tag{5.0.1a}$$

$$\psi(r, 0) = -2\frac{r}{\sigma^2}Ae^{-\frac{r^2}{\sigma^2}}, \tag{5.0.1b}$$

$$\pi(r, 0) = -\psi(r, 0) - \frac{\phi(r, 0)}{r}\left(1 - \frac{\breve{\beta}\breve{\gamma}}{\breve{\alpha}}\right), \tag{5.0.1c}$$

but since $\pi(r, 0)$ has a pole in $r = 0$ conditions proposed by Lora-Clavijo, Cruz-Osorio and Guzmán are not suitable for space-time without horizon. The choice of initial data does not influence the tail decay, therefore we choose simpler data.

Taylor series of functions ϕ, ψ and π show that they have certain kind of symmetry around $r = 0$:

1. For l even - ϕ and π are even and ψ is odd,

2. for l odd - ϕ and π are odd and ψ is even.

Symmetry relations are important in calculation estimated (here four-point) derivative of functions around $r = 0$.

The problem is accuracy - we used 10^{-15} accuracy in both programs. The bisection is using initial values with order of the magnitude 0, but still a full range of accuracy is used in computation.

We have not achieve our goal - the tail decay of scalar field ϕ, therefore it can not be examined whether it behaves like t^{-p}.

First try-outs on of numerical approach described in section 4.2 with simpler background showed proper behaviour (no explosion) of fields. The work is in progress next moths will probably give an answer to the question if ϕ approches zero in infinity like t^{-p}.

Chapter 6

Numerical methods used

In Chapter 6 the numerical methods used in the thesis will be briefly discussed. With Runge-Kutta method, the bisection method used in solving the system of equation (2.2.3) will be explained. The method of lines used to solve (4.1.19), for four-point radial derivative approximation and Runge-Kutta fourth-order method as a time integrator will be described. The Cubic spline interpolation, used in interpolating the background will be explained.

Numerical methods were based on those formed in [PTVF07].

6.1 Runge-Kutta method of solving differential equations

For ordinary differential equation given by:

$$\frac{df}{dx} = F(x, f), \tag{6.1.1}$$

we can calculate value of the function f_{i+1} in the point x_{i+1} from it's value f_i in x_i by moving on the tangent line given by the value of it's derivative:

$$f_{i+1} = f_i + (x_{i+1} - x_i)F(x_i, f_i). \tag{6.1.2}$$

Since it's using only value of derivative in starting point the first clue to achieve higher accuracy would be to use more of the points on the way (with $h = x_{i+1} - x_i$):

$$\begin{aligned} k_1 &= hF(x_i, f_i), \\ k_2 &= hF\left(x_i + \frac{h}{2}, f_i + \frac{k_1}{2}\right), \\ f_{i+1} &= f_i + k_2 + O(h^3). \end{aligned} \tag{6.1.3}$$

Expression above is second order Runge-Kutta method. In calculations we used fourth order Runde Kutta method, where step is calculated:

$$\begin{aligned} k_1 &= hF(x_i, f_i), \\ k_2 &= hF\left(x_i + \frac{h}{2}, f_i + \frac{k_1}{2}\right), \\ k_3 &= hF\left(x_i + \frac{h}{2}, f_i + \frac{k_2}{2}\right), \\ k_2 &= hF\left(x_i + h, f_i + k_3\right), \\ f_{i+1} &= f_i + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5). \end{aligned} \tag{6.1.4}$$

Shooting method is used to satisfy boundary value problems. In our case we want to assure $\phi_0(r)$ from (2.2.3) to be equal to 0 in infinity. We manipulate initial value of the function (in our case, since it's system of equations, one of initial values), to achieve that condition.

As it was shown in [FLP87], the system of equations (2.2.3) has a countable sequence of solutions satisfying the condition $\phi(r \to \infty) = 0$. We are interested in the one having no zeros (ϕ_0), thus we do bisection in two steps:

1. We search for solution having possibly small number of zeros, and the solution with close initial value, no zeros but not necessarily satisfying condition in infinity.

2. We bisect the interval between two initial values found in step 1, as long as accuracy of integrator allows us.

6.2 Line method of solving partial differential equations

We have partial differential equation given by:

$$\partial_{x_0} f = F(x_0, x_1, \ldots, x_n, \partial_{x_1} f, \ldots, \partial_{x_n} f), \tag{6.2.1}$$

for $n+1$ dimensional function f. We build the lattice in n dimensions - points $f_{i_1 \ldots i_n}$, which we evolve with ordinary integator (for example Runge-Kutta fouth-order method). The partial derivative of all but x_0 variables we estimate using m-points. For example first derivative in four-point method is given by:

$$\partial_x f = \frac{-f_{i-2} + 8f_{i-1} - 8f_{i+1} + f_{i+2}}{12}, \tag{6.2.2}$$

assuming 1+1 case (in more general more indexes in f appears, but all except one from derivative are not changed.

6.3 Cubic spline interpolation

Cubic Spline Interpolation is an extended version of linear interpolation assuring continuity of second derivative.

Let us assume we have function $f(x)$ tabulated as $f_i = f(x_i)$. Then the linear interpolation is given by:

$$f = Af_j + Bf_{j+1}, \tag{6.3.1}$$

where

$$\begin{aligned} x &\in [x_j, x_{j+1}], \\ A &= \frac{x_{j+1} - x}{x_{j+1} - x_j}, \\ B &= \frac{x - x_j}{x_{j+1} - x_j} = 1 - A. \end{aligned}$$

We extend it to:

$$f = Af_j + Bf_{j+1} + Cf_j'' + Df_{j+1}'', \tag{6.3.2}$$

where f_i'' are tabulated second derivatives, A and B are the same as for linear interpolation and

$$\begin{aligned} C &= \frac{1}{6}\left(A^3 - A\right)(x_{j+1} - x_j)^2, \\ D &= \frac{1}{6}\left(B^3 - B\right)(x_{j+1} - x_j)^2. \end{aligned}$$

We use the interpolation for finding even lattice of background functions, since their components calculated in non-compactified space-time will not be uniformly distributed in compactified space-time. We calculate second derivatives in non-compactified space-time using four-point estimation.

Appendix A

Taylor series coefficient for functions used

A.1 Boson Star equation

In (2.2.5) we has three functions: $\alpha(\hat{r})$, $a(\hat{r})$ and $\phi_0(\hat{r})$. Around $r = 0$ the Taylor series with $o(\hat{r}^{14})$ can be written as:

$$a(\hat{r}) = a_0 + a_1\hat{r} + a_2\hat{r}^2 + a_3\hat{r}^3 + a_4\hat{r}^4 + a_5\hat{r}^5 + a_6\hat{r}^6 + o(\hat{r}^8), \tag{A.1.1}$$

$$\alpha(\hat{r}) = \alpha_0 + \alpha_1\hat{r} + \alpha_2\hat{r}^2 + \alpha_3\hat{r}^3 + \alpha_4\hat{r}^4 + \alpha_5\hat{r}^5 + \alpha_6\hat{r}^6 + o(\hat{r}^8), \tag{A.1.2}$$

$$\phi_0(\hat{r}) = \varphi_0 + \varphi_1\hat{r} + \varphi_2\hat{r}^2 + \varphi_3\hat{r}^3 + \varphi_4\hat{r}^4 + \varphi_5\hat{r}^5 + \varphi_6\hat{r}^6 + o(\hat{r}^8). \tag{A.1.3}$$

To fulfil system (2.2.5) coefficient must be equal to:

$$a_{2i+1} = \alpha_{2i+1} = \varphi_{2i+1} = 0, i \in \{0, 1, 2, 3\}.$$

1. $a_0 = 1$
2. $a_2 = \frac{1}{3}(\frac{\varphi_0^2}{2} + \frac{\varphi_0^2}{2\alpha_0^2} + \frac{\Lambda\varphi_0^4}{4})$

3. $a_4 = \frac{1}{1440\alpha_0^4}(-32\varphi_0^2 - 32\alpha_0^2\varphi_0^2 + 64\alpha_0^4\varphi_0^2 - 36\varphi_0^4 + 168\alpha_0^2\varphi_0^4 + 60\alpha_0^4\varphi_0^4 - 32\alpha_0^2\Lambda\varphi_0^4 + 128\alpha_0^4\Lambda\varphi_0^4 + 84\alpha_0^2\Lambda\varphi_0^6 + 60\alpha_0^4\Lambda\varphi_0^6 + 64\alpha_0^4\Lambda^2\varphi_0^6 + 15\alpha_0^4\Lambda^2\varphi_0^8)$

4. $a_6 = \frac{1}{120960\alpha_0^6}(192\varphi_0^2 + 192\alpha_0^2\varphi_0^2 - 960\alpha_0^4\varphi_0^2 + 576\alpha_0^6\varphi_0^2 + 704\varphi_0^4 - 5024\alpha_0^2\varphi_0^4 + 1408\alpha_0^4\varphi_0^4 + 3488\alpha_0^6\varphi_0^4 + 768\alpha_0^2\Lambda\varphi_0^4 - 3648\alpha_0^4\Lambda\varphi_0^4 + 2880\alpha_0^6\Lambda\varphi_0^4 + 56\varphi_0^6 - 1080\alpha_0^2\varphi_0^6 + 7176\alpha_0^4\varphi_0^6 + 1400\alpha_0^6\varphi_0^6 - 3840\alpha_0^2\Lambda\varphi_0^6 + 3664\alpha_0^4\Lambda\varphi_0^6 + 8720\alpha_0^6\Lambda\varphi_0^6 - 2688\alpha_0^4\Lambda^2\varphi_0^6 + 4032\alpha_0^6\Lambda^2\varphi_0^6 - 540\alpha_0^2\Lambda\varphi_0^8 + 7176\alpha_0^4\Lambda\varphi_0^8 + 2100\alpha_0^6\Lambda\varphi_0^8 + 2256\alpha_0^4\Lambda^2\varphi_0^8 + 6976\alpha_0^6\Lambda^2\varphi_0^8 + 1728\alpha_0^6\Lambda^3\varphi_0^8 + 1794\alpha_0^4\Lambda^2\varphi_0^{10} + 1050\alpha_0^6\Lambda^2\varphi_0^{10} + 1744\alpha_0^6\Lambda^3\varphi_0^{10} + 175\alpha_0^6\Lambda^3\varphi_0^{12})$

1. $\alpha_0 = \alpha_0$

2. $\alpha_2 = \frac{4\varphi_0^2 - 2\alpha_0^2\varphi_0^2 - \alpha_0^2\Lambda\varphi_0^4}{12\alpha_0}$

3. $\alpha_4 = \frac{1}{1440\alpha_0^3}(-48\varphi_0^2 + 72\alpha_0^2\varphi_0^2 - 24\alpha_0^4\varphi_0^2 + 16\varphi_0^4 + 32\alpha_0^2\varphi_0^4 - 20\alpha_0^4\varphi_0^4 + 72\alpha_0^2\Lambda\varphi_0^4 - 48\alpha_0^4\Lambda\varphi_0^4 + 16\alpha_0^2\Lambda\varphi_0^6 - 20\alpha_0^4\Lambda\varphi_0^6 - 24\alpha_0^4\Lambda^2\varphi_0^6 - 5\alpha_0^4\Lambda^2\varphi_0^8)$

4. $\alpha_6 = \frac{1}{362880\alpha_0^5}(768\varphi_0^2 - 1920\alpha_0^2\varphi_0^2 + 1536\alpha_0^4\varphi_0^2 - 384\alpha_0^6\varphi_0^2 - 320\varphi_0^4 - 3520\alpha_0^2\varphi_0^4 + 6416\alpha_0^4\varphi_0^4 - 2288\alpha_0^6\varphi_0^4 - 2976\alpha_0^2\Lambda\varphi_0^4 + 4896\alpha_0^4\Lambda\varphi_0^4 - 1920\alpha_0^6\Lambda\varphi_0^4 - 1344\varphi_0^6 + 3744\alpha_0^2\varphi_0^6 + 144\alpha_0^4\varphi_0^6 - 840\alpha_0^6\varphi_0^6 - 1584\alpha_0^2\Lambda\varphi_0^6 + 9896\alpha_0^4\Lambda\varphi_0^6 - 5720\alpha_0^6\Lambda\varphi_0^6 + 3360\alpha_0^4\Lambda^2\varphi_0^6 - 2688\alpha_0^6\Lambda^2\varphi_0^6 + 1872\alpha_0^2\Lambda\varphi_0^8 + 144\alpha_0^4\Lambda\varphi_0^8 - 1260\alpha_0^6\Lambda\varphi_0^8 + 3480\alpha_0^4\Lambda^2\varphi_0^8 - 4576\alpha_0^6\Lambda^2\varphi_0^8 - 1152\alpha_0^6\Lambda^3\varphi_0^8 + 36\alpha_0^4\Lambda^2\varphi_0^{10} - 630\alpha_0^6\Lambda^2\varphi_0^{10} - 1144\alpha_0^6\Lambda^3\varphi_0^{10} - 105\alpha_0^6\Lambda^3\varphi_0^{12})$

1. $\varphi_0 = \varphi_0$

2. $\varphi_2 = \frac{-\varphi_0 + \alpha_0^2\varphi_0 + \alpha_0^2\Lambda\varphi_0^3}{6\alpha_0^2}$

3. $\varphi_4 = \frac{1}{360\alpha_0^4}(3\varphi_0 - 6\alpha_0^2\varphi_0 + 3\alpha_0^4\varphi_0 + 8\varphi_0^3 - 12\alpha_0^2\varphi_0^3 + 10\alpha_0^4\varphi_0^3 - 12\alpha_0^2\Lambda\varphi_0^3 + 12\alpha_0^4\Lambda\varphi_0^3 - 4\alpha_0^2\Lambda\varphi_0^5 + 15\alpha_0^4\Lambda\varphi_0^5 + 9\alpha_0^4\Lambda^2\varphi_0^5 + 5\alpha_0^4\Lambda^2\varphi_0^7)$

4. $\varphi_6 = \frac{1}{45360\alpha_0^6}(-9\varphi_0 + 27\alpha_0^2\varphi_0 - 27\alpha_0^4\varphi_0 + 9\alpha_0^6\varphi_0 - 136\varphi_0^3 + 456\alpha_0^2\varphi_0^3 - 618\alpha_0^4\varphi_0^3 + 298\alpha_0^6\varphi_0^3 + 153\alpha_0^2\Lambda\varphi_0^3 - 306\alpha_0^4\Lambda\varphi_0^3 + 153\alpha_0^6\Lambda\varphi_0^3 - 112\varphi_0^5 + 296\alpha_0^2\varphi_0^5 - 248\alpha_0^4\varphi_0^5 + 280\alpha_0^6\varphi_0^5 + 312\alpha_0^2\Lambda\varphi_0^5 - 1227\alpha_0^4\Lambda\varphi_0^5 + 1065\alpha_0^6\Lambda\varphi_0^5 - 315\alpha_0^4\Lambda^2\varphi_0^5 + 315\alpha_0^6\Lambda^2\varphi_0^5 + 132\alpha_0^2\Lambda\varphi_0^7 - 88\alpha_0^4\Lambda\varphi_0^7 + 560\alpha_0^6\Lambda\varphi_0^7 - 477\alpha_0^4\Lambda^2\varphi_0^7 +$

$1122\alpha_0^6\Lambda^2\varphi_0^7 + 171\alpha_0^6\Lambda^3\varphi_0^7 + 18\alpha_0^4\Lambda^2\varphi_0^9 + 350\alpha_0^6\Lambda^2\varphi_0^9 + 355\alpha_0^6\Lambda^3\varphi_0^9 + 70\alpha_0^6\Lambda^3\varphi_0^{11})$

Significant influence on numerical method has F_0 (where F is any of functions above) as its value in $r = 0$ and $F_1 = 0$ as value of its derivative in $r = 0$ (since equations contain r^{-1} and r^{-2} terms). Since to numerically compute system (2.2.5) we introduce new function $\xi(\hat{r}) = \partial_{\hat{r}}\varphi_0(\hat{r})$ important will be $\varphi_1 = 0$ and $2\varphi_2$ as value of ξ and its derivative in $r = 0$.

Calculating 7 coefficients to each function seems to be waste of time, but it is necessary to study behaviour of solution to wave equation (4.1.19) (and with good accuracy).

A.2 Heights function

Taylor series of heights function's derivative has a point in finding it's value in $r = 0$, then:

$$\frac{dh}{d\hat{r}} = h'(\hat{r}) = h_0 + h_1\hat{r} + h_2\hat{r}^2 + o(\hat{r}^3). \tag{A.2.1}$$

$C \neq 0$

1. $h_0 = -\frac{1}{\alpha_0}$,
2. $h_1 = 0$,
3. $h_2 = \frac{\varphi_0^2+k^2\varphi_0^2-2\alpha_0^2\varphi_0^2-2k^2\alpha_0^2\varphi_0^2-\alpha_0^2\Lambda\varphi_0^4-k^2\alpha_0^2\Lambda\varphi_0^4}{6\sqrt{(1+k^2)^2}\alpha_0^3}$.

$C = 0$

1. $\bar{h}_0 = 0$,
2. $\bar{h}_1 = \frac{k}{12\alpha(0)}$,
3. $\bar{h}_2 = 0$,
4. $\bar{h}_3 = -\frac{\left(\left(2880k\alpha(0)^2+5k^3\alpha(0)^2+288k\varphi_0(0)^2-720k\alpha(0)^2\varphi_0(0)^2-360k\alpha(0)^2\Lambda\varphi_0(0)^4\right)\right)}{17280\alpha(0)^3}$,

5. $\bar{h}_4 = 0$.

Appendix B

Time dependence of field Φ

Lagrangian density given in (2.1.1) is invariant under the global gauge transformation, the conserved current j^μ implied by Noether theorem is given by:

$$j^\mu = -i\left(\Phi^*\partial_\nu\Phi - \partial_\nu\Phi^*\Phi\right)g^{\nu\mu}. \tag{B.0.1}$$

Number of particles is given by $N = \int d^3x j^t\sqrt{-g}$. We need to find time coordinate for the current, assuming $g^{t\mu} = 0$, $\forall\mu \neq t$:

$$j^t = -i\left(\Phi^*\partial_t\Phi - \partial_t\Phi^*\Phi\right)g^{tt}. \tag{B.0.2}$$

Additionally splitting the complex field Φ into two real fields:

$$\Phi = \frac{\Phi_R + i\Phi_I}{\sqrt{2}}, \tag{B.0.3}$$

we obtain:

$$N = -i\int d^3x\left(\Phi_R\partial_t\Phi_I - \partial_t\Phi_R\Phi_I\right)g^{tt}\sqrt{-g}. \tag{B.0.4}$$

Minimising energy of the system we obtain (following [FLP87]):

$$\delta\left(E + \omega N\right) = 0 \tag{B.0.5}$$

where ω is Lagrange multiplier. Direct calculation leads to the system of equations:

$$\partial_t \Phi_R = \omega \Phi_I, \tag{B.0.6a}$$

$$\partial_t \Phi_I = -\omega \Phi_R, \tag{B.0.6b}$$

that can be satisfy by:

$$\Phi(\vec{x}, t) = \phi(\vec{x}) e^{-i\omega t}. \tag{B.0.7}$$

Applying spherical symmetry will simplify $\phi(\vec{x}) \to \phi(r)$.

Appendix C

List of attachments

Part or this thesis are as well as above text:

1. Program *BosonStarMetricCompac* (source: bosonstar.f95) computing:

    - solution of set of equations (2.2.5) with respect of parameters Λ (defined in (2.2.4e)) and $\phi_0(0)$ - output file: 'bosonS.dat',
    - numerical values of $h'(\hat{r})$ (defined in (3.3.19)) - output file: 'height.dat',
    - numerical values of Ricci scalar for compactified space-time () - output file: 'RicciS.dat',
    - numerical values of functions $\breve{\alpha}(r)$, $\breve{\beta}(r)$, $\breve{\gamma}(r)$ (defined in (4.1.1)) - output file : 'backgr.dat',
    - intepolation of Ricci scalar and $\breve{\alpha}(r)$, $\breve{\beta}(r)$, $\breve{\gamma}(r)$ functions on even lattice, prepared as an input file for program solving wave equation - output file : 'output.dat'.

Appendix D

Bibliography

Bibliography

[BW00] P. Bizoń and A. Wasserman. On existence of mini-boson stars. *Communications in Mathematical Physics*, 215:357–373, 2000.

[BZ09] P. Bizoń and A. Zenginoğlu. Universality of global dynamics for the cubic wave equation. *Nonlinearity*, 22:2473–2485, 2009.

[EGA+05] F. Eisenhauer, R. Genzel, T. Alexander, Abuter R., Paumard T., T. Ott, A. Gilbert, S. Gillessen, F. Horrobin, S. Trippe, H. Bonnet, C. Dumas, A Kaufer, M. Kissler-Patig, and G. Monnet. Sinfoni in the galactic center: Young stars and infrared flares in the central light-month. *The Astrophysical Journal*, 628:246–259, Jul 2005.

[FLP87] R. Friedberg, T. D. Lee, and Y. Pang. Mini-soliton stars. *Phys. Rev. D*, 35(12):3640–3657, Jun 1987.

[Gle88] Marcelo Gleiser. Stability of boson stars. *Phys. Rev. D*, 38(8):2376–2385, Oct 1988.

[Gou07] Éric Gourgoulhon. 3+1 formalism and bases of numerical relativity. arXiv:gr-qc/0703035v1, 2007.

[Kau68] David J. Kaup. Klein-gordon geon. *Phys. Rev.*, 172(5):1331–1342, Aug 1968.

[LCCOG10a] F. D. Lora-Clavijo, A. Cruz-Osorio, and F. S. Guzmán. Evolution of a massless test scalar field on boson star space-times. *Phys. Rev. D*, 82(2):023005, Jul 2010.

[LCCOG10b] F. D. Lora-Clavijo, A. Cruz-Osorio, and F. S. Guzmán. Universality of global dynamics for the cubic wave equation. arXiv:1007.1162v2 [gr-qc], 2010.

[PTVF07] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery. *Numerical Recipes - The Art of Scientific Computing*. Cambridge University Press, third edition, 2007.

[RB69] Remo Ruffini and Silvano Bonazzola. Systems of self-gravitating particles in general relativity and the concept of an equation of state. *Phys. Rev.*, 187(5):1767–1783, Nov 1969.

[Zen10] Anıl Zenginoğlu. Hyperboloidal layers for hyperbolic equations on unbounded domains. *Classical and Quantum Gravity*, arXiv:1008.3809v2 [math.NA], 2010.